On the APPLICATION of a CONVERGING SERIES to the CONSTRUCTION of LOGARITHMS. By Mr. WILLIAM ALLMAN, A. B. *Trinity College, Dublin.*

FROM a due consideration of Newton's binomial theorem, it may be shewn, that the logarithm of the ratio of one number to another, according to Napier's system, is equal to the sum of the series, $\frac{2d}{s}+\frac{2d^3}{3s^3}+\frac{2d^5}{5s^5}+\frac{2d^7}{7s^7}+$ &c. d representing the difference, and s the sum of the numbers: which logarithm of the ratio, added to, or subtracted from, the logarithm of the antecedent, according as the antecedent is less, or greater, than the consequent, gives the logarithm of the consequent. Read May 21, 1796.

In any system whatever, the logarithm of the ratio of one number to another is equal to the sum of this series, $\frac{2pd}{s}+\frac{2pd^3}{3s^3}+\frac{2pd^5}{5s^5}+\frac{2pd^7}{7s^7}+$ &c. where d expresses the difference,

and

and s the fum of the numbers, as before; and p the quote found upon dividing, the logarithm of fome number according to that fyftem, by Napier's logarithm of the fame number.

It is evident, that the lefs d is in refpect of s, the fafter the feries will converge; fo that the conftruction of the logarithms of prime numbers, will be rendered more eafy and expeditious, by finding two great products, which fhall have a fmall difference; one of which products, fhall be compofed entirely of factors whofe logarithms are already known, and the other, fhall have in its compofition, the number whofe logarithm is fought, or fome power of that number; and, if it have any other factors, the logarithms of thefe factors muft be previoufly known.

Having found fuch products, we may, by the application of the above-mentioned feries, find the logarithm of their ratio to each other; which is the fame with the logarithm of the ratio of the firft product (or that which is compofed entirely of factors whofe logarithms are known) divided by the factor or compound of factors whofe logarithms are known (if there be any fuch) in the latter product, to the prime number whofe logarithm is fought, or fome power of that number. Then, from the logarithm of the antecedent, and the logarithm of the ratio, we have, by addition or fubtraction, the logarithm of the confequent.

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I PROPOSE now to ſhew, how products of this nature may be found—and firſt, when they riſe to two dimenſions only, or conſiſt each of two factors.

LET the two factors of one product be $x+a$, and $x+b$; that product will then be $x^2 \begin{matrix}+a\\+b\end{matrix} x + ab$; if we aſſume for two other factors, the quantities x and $x \begin{matrix}+a\\+b\end{matrix}$ we get the product $x^2 \begin{matrix}+a\\+b\end{matrix} x$, which differs from the firſt product by ab, the loweſt term of that product.—Here it is obvious,

1. THAT the nearer ab approaches to o, cæteris paribus, the faſter will the ſeries converge.

2. THAT if $a+b$ be made equal to o there will be but three different factors, for $x \begin{matrix}+a\\+b\end{matrix}$ will then be equal to x.

3. IT may be obſerved with reſpect to products of any dimenſions whatever, that the ſame factor muſt not enter both products, for this would ſerve only to raiſe the terms of the fraction $\frac{d}{s}$; not to diminiſh its value.

4. HENCE it follows, that in ſearching for theſe products, we muſt ſuppoſe one of them to conſiſt entirely of compound factors.

5. It is convenient that they differ in their loweſt terms only; otherwiſe the quote found upon dividing their ſum by their difference (by the increaſe or decreaſe of which the convergency of the ſeries is accelerated or retarded) would be of a lower dimenſion than either product; ſo that we would be at the trouble of finding products of a higher dimenſion, when perhaps products might be found of the ſame dimenſion with the quote $\frac{s}{d}$, differing only in their loweſt terms.

Besides, if this compound difference (or ſome compound quantity which is of the ſame dimenſion with this difference and an aliquot part of it) do not meaſure either product, there is this additional inconvenience, namely, that the numerator of the fraction is uncertain, being affected by the variable quantity x; whereas if the difference of the products be the ſame with the difference of their abſolute terms, it is in no manner affected by the variation of x, and the numerator of the fraction $\frac{d}{s}$ may ſometimes be reduced to unity, which is convenient in the application of the ſeries to practice.

If the compound difference (or ſome aliquot part of it, and of the ſame dimenſions with it) meaſure either product, it is a ſure ſign that there are products of an inferior dimenſion every whit as advantageous for the conſtruction of logarithms. From all

all which we may fee the utility of fo adapting the products to each other, that they may differ in their loweft terms only.

6. It is ufeful, when one of the products only confifts entirely of compound factors, that the fecond terms of thefe factors be as fmall numbers as poffible; for the continual product of all thefe fecond terms conftitutes in this cafe the difference of the products. This precaution is alfo ufeful, when both products confift entirely of compound factors, and their loweft terms are, at the fame time, affected with contrary figns. For the difference of the products in this cafe is equal to the fum of two numbers which are produced by the continual multiplication of the fecond terms of the factors of each product refpectively, no regard being paid to their figns.

Substitute in the above products of two dimenfions, 1 for a, and -1 for b (by which means we fhall have $ab = -1$, and $a + b = o$) the products will then be $x^2 - 1$, and x^2, whofe difference is unity and which confift of but three different factors $x - 1$, $x + 1$ and x.

The logarithms of any two of thefe factors being given that of the third may be found by the application of the above mentioned feries; $\frac{d}{s}$ being in this cafe equal to $\frac{1}{2x^2 - 1}$.

But Dr. Halley has already ſufficiently explained this method of applying the ſeries.

To come then in the next place to products of three dimenſions, or which conſiſt each of three factors—

Let x^3+qx^2+rx+s repreſent one of ſuch products, and let its factors be denoted by $x+a$, $x+b$ and $x+c$. Then if the ſign of the ſecond term in each factor be changed, the ſign of the firſt term in each being unvaried, the product will be x^3-qx^2+rx-s; make q, or $a+b+c=o$; we then ſhall have two products differing from each other by $2s=2abc$; ſo that if c be taken equal to $a+b$ with their ſigns changed, the products will differ from each other in their loweſt terms only; but this difference, and at the ſame time the number of different factors, will be the leaſt poſſible (fractional numbers being ſet aſide) if we make $a=b$ and each of them equal to unity; the products then with their reſpective factors will be, $x^3 * -3x-2=\overline{x+1}|^2\times x-2$ and $x^3 * -3x+2=\overline{x-1}|^2\times x+2$; whoſe difference never exceeds four in whatever manner x be varied; and the number of different factors is alſo reduced to four.

That the number of different factors of ſuch products as are required of three dimenſions, cannot be leſs than four, may be thus demonſtrated; if there were but three different factors one

one product ſhould have all its factors equal to each other; the other, two of the three factors equal to each other. Let $x^3 + 3ax^2 + 3a^2x + a^3$ ($= \overline{x+a}|^3$) repreſent one of the products, and $x^3 \begin{matrix} +2b \\ +c \end{matrix} x^2 \begin{matrix} +b^2 \\ +2bc \end{matrix} x + b^2c$ ($= \overline{x+b}|^2 \times \overline{x+c}$) the other. Then ſince theſe products ought to differ in their loweſt terms only, we have $3a = 2b + c$, and $b^2 + 2bc = 3a^2$, by comparing theſe equations with each other we find $4b^2 + 4bc + c^2 = 3b^2 + 6bc$ $\because b^2 = 2bc - s^2$ and $b = c$; ſo that theſe products will be both cubes, or can have but two different factors; but ſince $3a = 2b + c$, i. e. (from what has been already proved) $3a = 3b$, we have $a = b$; by which it appears that not only the ſame factor enters both products (which has been already ſhewn to be uſeleſs) but both products conſiſt entirely of the ſame factor equally repeated and are in effect *one* and the ſame product.

In the application of the products $x^3 - 3x - 2$ and $x^3 - 3x + 2$, to the conſtruction of logarithms the fraction $\frac{d}{s}$ of the ſeries muſt be made equal to $\frac{4}{2 \times \overline{x+1}|^2 \times \overline{x-2} + 4}$ or $\frac{4}{2 \times \overline{x-1}|^2 \times \overline{x+2} - 4}$ or in its loweſt terms $\frac{1}{\frac{\overline{x+1}|^2 \times \overline{x-2}}{2} + 1}$ or $\frac{1}{\frac{\overline{x-1}|^2 \times \overline{x+2}}{2} - 1}$.

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The ſeries will then exhibit the logarithm of the ratio of $\overline{x+1}|^2 \times \frac{x-2}{x+2} : \overline{x-1}|^2$, or of $\overline{\frac{x+1}{x-1}}|^2 \times \overline{x-2} : x+2$; of $\overline{x+1}|^2 : \overline{x-1}|^2 \times \frac{x+2}{x-2}$, or of $x-2 : \overline{\frac{x-1}{x+1}}|^2 \times \overline{x-2}$; where, the antecedents being leſs than the conſequents, if the logarithm of $\overline{x+1}|^2$ or of $x-2$ be required, the ſum of the above ſeries is to be ſubtracted from the logarithm of $\overline{x-1}|^2 \times \frac{x+2}{x-2}$ or of $\overline{\frac{x-1}{x+1}}|^2 \times x+2$, reſpectively.

The ſmaller the factor, by which any given number whoſe logarithm is ſought, is denoted, the faſter will the ſeries converge.

For the abſolute value of each factor, and therefore the value of each product, is by this means encreaſed.—Then ſince the ſum of the products is increaſed, their difference being unvaried, the ſeries will converge faſter.

Though we have this advantage by making a ſmaller factor repreſent the number whoſe logarithm is required, yet it may be objected, that this implies the neceſſity of being previouſly acquainted with the logarithms of greater numbers. But the difficulty of this, will in the preſent caſe, be removed, if the prime number whoſe logarithm is required (that number being underſtood

understood to exceed 3) be denoted by the factor $x-1$, when the number next less is measured by 3; or by the factor $x-2$, when the number next greater is measured by 3.

Of the four different factors, two are always even and two odd, the number whose logarithm is sought being odd, there *can* be but one odd number greater than the given one, whose logarithm it is necessary to be previously acquainted with, in order to find the logarithm required. This number exceeds the given one by 2, when the given one is denoted by $x-1$; by 4 when it is denoted by $x-2$; so that this odd factor, greater than that which represents the number whose logarithm is sought, will be measured by 3, if the given number be represented by $x-1$, and at the same time, the number next less be measured by 3; or if the given number be represented by $x-2$, and the number next greater be measured by 3; we then shall have all the factors, except the number whose logarithm is sought, composite. That their component parts will be less than that number, is evident from the nature of the factors.

If any one wishes rather constantly to use the same notation, which may perhaps be desirable, in constructing a table of logarithms, for the sake of avoiding confusion; when a prime factor occurs greater than that whose logarithm is sought, so that the logarithm cannot be found immediately, let the logarithm

of

of that prime number be sought by the same method as that which was to be used for the discovery of the first logarithm; and no prime factor will occur to prevent the logarithm of this number from being found immediately.—For, since, when the factor $x-1$ is taken to denote the prime number whose logarithm is sought, the other odd factor exceeds it by 2; and when the factor $x-2$ is taken to denote it, the remaining odd factor exceeds it by 4; if the number exceeding the given one, when denoted by the factor $x-1$, by two, or when denoted by the factor $x-2$, by 4, be composite, the logarithm may be found immediately: But if, in the first case, the number exceeding the given one by 2, be prime, the number which exceeds that number by 2 (or the given one by 4) will be composite—and if in the other case, the number exceeding the given one by 4 be prime, the number which exceeds that number by 4 (or the given one by 8) will be composite—so that the method of notation remaining unvaried, no prime factor will occur to prevent the logarithm of that prime number (by the intervention of which, the logarithm first sought is to be deduced) from being found immediately.

It is easy to shew, that, if the number which exceeds any prime number greater than 3, by 2, be prime, the number which exceeds it by 4, will be composite; or if the number which exceeds it by 4 be prime, that number which exceeds it

it by 8 will be compofite; for, fince neither the given number, nor the number which exceeds it by 2, is meafured by 3, according to the firft fuppofition, the number which exceeds the given one by unity, will be meafured by 3 (for 3 muft neceffarily meafure one of three fucceffive numbers) and therefore the number exceeding the given one by 4 will be meafured by 3. Since, in the fecond place, either the number which exceeds the given one by 2, or that which exceeds it by 4 is meafured by 3; if the number which exceeds it by 4 be prime, that which exceeds it by 2, and therefore, that alfo which exceeds it by 8, will be meafured by 3.

It were eafy to fhew, that, in the application of the above products of three dimenfions, differing by 4, to the conftruction of logarithms, the numerator of the fraction $\frac{d}{s}$ reduced to its loweft terms is always unity. In this cafe $\frac{d}{s} = \frac{2}{\overline{x+1}|^2 \times \overline{x-2} + 2}$. Now, fince either $\overline{x+1}|^2$, or $x-2$, is even, 2 will meafure their product increafed by 2.

Make $\frac{\overline{x+1}|^2 \times \overline{x-2}}{2} + 1 = y$; the general feries will then be reduced to this form, $\frac{2p}{y} + \frac{2p}{3y^3} + \frac{2p}{5y5} + \frac{2p}{7y7} +$, &c. Half the

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ſum of this ſeries will expreſs the logarithm of the ratio of $\overline{x+1} \times \overline{\frac{x-2}{x+2}}\Big|^{\frac{1}{2}} : x-1$ or of $x+1 : \overline{x-1} \times \overline{\frac{x+2}{x-2}}^{\frac{1}{2}}$

If two products of any dimenſions whatever, differ in their loweſt terms only, and the ſecond terms of all the factors in each product be equally multiplied; the products of the factors ſo changed will ſtill differ in their loweſt terms only, but the difference in this caſe will be the difference of the original products multiplied into the common multiplier of the ſecond terms of the factors raiſed to the ſame dimenſion with the higheſt term of either product. For the terms of the original products, beginning with the higheſt term in each, are reſpectively multiplied by the terms of a geometrical progreſſion, whoſe firſt term is unity, and ſecond the common multiplier; ſo that the terms of the latter products, which are correſpondent to terms that were equal in the firſt products, will be equal to each other; and the terms of the latter, correſpondent to thoſe which differed in the firſt products, will have a difference equal to the difference of the terms in the firſt products, multiplied by the correſpondent term of the geometrical progreſſion above-mentioned.

Let m denote the common multiplier, and n the index of the higheſt term in the products; the geometrical progreſſion will ſtand thus, $1, m, m^2, m^3$, &c. , m^n. Let the latter products be both

both divided by m^n; the quotes will have the fame ratio as the latter products, and the fame difference as the firft products have. Now the terms of the firft products multiplied by the correfpondent terms of this progreffion, $\frac{1}{m^n}, \frac{1}{m^{n-1}}, \frac{1}{m^{n-2}}, \frac{1}{m^{n-3}}$ &c. $\frac{1}{m^3}, \frac{1}{m^2}, \frac{1}{m^1}$, 1 (i. e. the terms of the preceding progreffion divided by m^n) gives the correfpondent terms of the quotes; fo that if m be greater than 1, the terms of the quotes will be lefs than the terms of the firft products refpectively.—The contrary to this happens, if m be a proper fraction—In this laft cafe then, the fum of the quotes being greater than that of the firft products, while the difference is the fame; and the difference of the quotes having the fame ratio to their fum, as the difference of the latter products to their fum, it appears that the difference of the latter products bears a lefs ratio to their fum, than the difference of the firft products bears to their fum; fo that if the fecond terms of the factors of products differing in their loweft terms only, be all multiplied by the fame proper fraction, the feries expreffing the logarithm of the ratio of the products to each other will converge fafter.

If the fecond terms of the factors of products differing in their loweft terms only, be equally increafed by addition or

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diminiſhed by ſubtraction, the products will ſtill have the ſame difference.

Let the coefficients of the 2d, 3d, 4th, 5th, &c. terms of the given products be denoted by q, r, s, t, v, &c. reſpectively; the index of the higheſt term by w, and the quantity, whether affirmative or negative, which is to be added to each of the ſecond terms of the factors, by m, the coefficients of the correſpondent terms of the new products will be, $\frac{n}{1}.m$, $\frac{n}{1}.\frac{n-1}{2}.m^2$, $\frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}.m^3$

$$\begin{array}{llll}
\frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}m^4, \text{ \&c.} & q & \frac{n-1}{1}qm & \frac{n-1}{1}.\frac{n-2}{2}qm^2 \\
\frac{n-1}{1}.\frac{n-2}{2}.\frac{n-3}{3}qm^3 & & r & \frac{n-2}{1}rm \\
\frac{n-2}{1}.\frac{n-3}{2}rm^2 & & & s \\
\frac{n-3}{1}sm & & & \\
t. & & &
\end{array}$$

Whence it appears that in whatſoever term the given products begin to differ from each other in the correſpondent one, will the new products alſo begin to differ from each other, and by the ſame quantity. If the ſecond terms of the factors be *equally* increaſed, the ſum of the products will be increaſed, ſo that (if

(if the abſolute value of the leading term be unvaried) the ſeries will converge faſter. If then the given number whoſe logarithm is ſought, be denoted by a ſimple quantity x, the lower the place which the ſimple quantity occupies among the factors, or the fewer reſidual factors there are, the faſter will the ſeries converge.—Now whatever compound quantity be ſubſtituted for this ſimple one, the effect will be the ſame, viz. that the convergency of the ſeries is quicker or ſlower, the lower or higher the place which the given number occupies in the rank of factors.

But ſince it is uſeful that the difference of the greateſt and leaſt factor ſhould be as ſmall as poſſible (as follows from what has been obſerved before) the advantage of a ſwifter convergency, will, generally ſpeaking, be in a leſs degree in this caſe, than the ſimilar advantage which ariſes from multiplying the ſecond term of each factor by an aliquot fraction, or (which amounts to the ſame thing) the leading term of each factor by the reciprocal integer.

From what has been ſaid it follows, that if, of the firſt and ſecond, fourth and fifth terms of an arithmetical progreſſion, the logarithms of any three be given, the logarithm of the remaining term may be found. If the common difference of the progreſſion be unity, $\frac{d}{s}$, will be equal to an aliquot fraction, whoſe denominator

denominater exceeds, by 1, half the product, whofe factors are the firft or leaft term of the progreffion, and the fquare of the fourth term; if the common difference be any other number, fuppofe m; $\frac{d}{s}$ will be equal to that aliquot fraction above defcribed, multiplied by m^3. The general feries will then exprefs the logarithm of the ratio which the product of the firft term into the fquare of the fourth, bears to the product of the fifth term into the fquare of the fecond. This, fince the arithmetical progreffion is an increafing one, will be a ratio of leffer inequality.

It alfo appears, that there is no reftriction fet on finding the moft convenient products, by the fuppofition that one of them is defective in its loweft term. And that, in the inveftigation of products, if we find fecond terms which (all of them being equally diminifhed fo that one may vanifh) will then admit a common meafure, the products may then be reduced to a fmaller difference.

Let $x^4 + qx^3 + rx^2 + sx + t$ reprefent a product of four dimenfions, and let s be $= o$, two factors are to be found, whofe product fhall be $x^2 + qx + r$; this being multiplied by x^2 will give the product $x^4 + qx^3 + rx^2$, differing from the firft product by t the loweft term

term of that product. Suppose one of the factors to be $x+m$; the other will be $x \overset{+q}{-} m$. Then $qm - m^2 = r \therefore m = \frac{q}{2} \mp \sqrt{\frac{q^2}{4} - r}$, and $q - m = \frac{q}{2} \mp \sqrt{\frac{q^2}{4} - r}$. Let the factors of the product $x^4 + qx^3 + rx^2 + sx + t$ be $x+a$, $x+b$, $x+c$, and $x+d$. And the factors of the product $x^2 + qx + r$ will be, $x + \frac{a+b+c+d}{2} \pm \sqrt{\frac{(a+b+c+d)^2}{4} - ab - ac - ad - bc - bd - cd}$, or (which is the same thing) $x + \frac{a+b+c+d}{2} \pm \sqrt{\frac{(a+b-c-d)^2}{4} - ab - cd}$. These factors will be always rational (whatever may be the values of a, b, c, and d, provided they be rational) if we make $ab + cd = 0$. Then since s, or $abc + abd + acd + bcd = 0$, we have $\frac{cd}{b}\ (= -a) = \frac{bcd}{bc + bd + cd}$. And dividing by cd and multiplying by the denominators, $bc + bd + cd = b^2$. Therefore $c = \frac{b^2 - bd}{b+d}$, $d = \frac{b^2 - bc}{b+c}$ and $b = \frac{c+d}{2} \pm \sqrt{\frac{(c+d)^2}{4} + cd}$. So that if b be rational, c, d, and a (or $-\frac{cd}{b}$) will also be rational. But b will be rational, if $c^2 + 6cd + d^2$ be a perfect square. Make $c^2 + 6cd + d^2 = c^2$. Then $6cd + d^2 = 0$ and $d = -6c$. Universally, putting $c^2 + 6cd + d^2 = c^2 + 2cn + n^2$, we have $\overline{6d - 2n}.\ c = n^2 - d^2$

and

and $c = \frac{n^2 - d^2}{6d - 2n}$. Only we muſt not aſſume $n = d$ or $= - d$; or $= 3d$, for on the two firſt aſſumptions we ſhould get $\overline{6d - 2n}$. $c = o$, i. e. either $4dc$, or $8dc = o$; on the third aſſumption, $n^2 - d^2 = o$, i. e. $8d^2 = o$; ſo that either c or d would be equal to o; contrary to what has been above laid down.

MAKE $c = - 1$ and $d = 6$; then $b = 2$ or 3, let $b = 2$; and we ſhall have $a = 3$. And m, as alſo $q - m$, $= 5$.

THEN we have the factors $x - 1$, $x + 2$, $x + 3$, $x + 6$, whoſe product ſhall be deficient in its penultimate term, and ſhall differ by 36 its loweſt term, from the product of the factors x^2 and $\overline{x + 5}^2$; as will appear by their multiplication.

$$\begin{array}{l}
x + 6 \\
x + 3 \\
\hline
x^2 + 9x + 18 \\
x + 2 \\
\hline
x^3 + 11x^2 + 36x + 36 \\
x - 1 \\
\hline
x^4 + 10x^3 + 25x^2 * - 36
\end{array}
\qquad\qquad
\begin{array}{l}
x + 5 \\
x + 5 \\
\hline
x^2 + 10x + 25 \\
x^2 \\
\hline
x^4 + 10x^3 + 25x^2
\end{array}$$

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In the ſecond product, we may obſerve that there are but two different factors. That the factor $x+m$ muſt be equal to the factor $x\begin{smallmatrix}+q\\-m\end{smallmatrix}$, or that m is equal to $\frac{q}{2}$, will appear if it be ſhewn that $\frac{q^2}{4}-r$ or $\frac{\overline{a+b-c-d}|^2}{4}-ab-cd=o$; which may be done in the following manner:—Since $ab+cd=o$, we alſo have by multiplication, $a^2b+acd=o$, $ab^2+bcd=o$, $abc+c^2d=o$, and $abd+cd^2=o$.

By comparing theſe four equations ſeverally with the equation, $abc+abd+acd+bcd=o$, we find,

$$\begin{aligned} a^2 &= ac+ad+cd \\ b^2 &= bc+bd+cd \\ c^2 &= ab+ac+bc \\ d^2 &= ab+ad+bd \\ \hline \end{aligned}$$

$a^2+b^2+c^2+d^2=2ab+2ac+2ad+2bc+2bd+2cd$. And therefore $\overline{a+b-c-d}|^2=4ab+4cd \therefore \frac{\overline{a+b-c-d}|^2}{4}-ab-cd=o$. Q. E. D.

That the number of different factors in both products cannot be leſs than ſix, without introducing ſurds, may be ſhewn as follows:

First, neither product can have three factors alike—Since, by a change in all the second terms, any factor may be reduced to a simple expression; let the factor which occurs three times be denoted by x, the product will then be deficient in the three lowest terms—the other product that it may differ from this by its lowest term only, must want the penultimate and antepenultimate terms; at the same time, retaining the last; so that two of its factors will be impossible.

The above is evidently applicable to products of all dimensions; and from this immediately follows what has been proved above, relative to the least number of different factors in products of three dimensions.

Hence it appears that the least possible number of different factors will not be less than the index of the highest term in either product, if that index be even—if it be odd, the least possible number of different products will be greater than the index.

Whence it follows that the number of different factors in the present case cannot possibly be less than four.

But in the next place, the number of different factors cannot without introducing surds, be less than six.

For

For if we suppose but five different factors in both products, two of them must be in one product, and three in the other, since neither product can consist entirely of equal factors, as appears from the above, and neither of those two factors must be taken three times. Let the factors of one product then be be x^2 and $\overline{x+m}|^2$, and the factors of the other $x+a$, $x+b$, $\overline{x+c}|^2$. The respective products will be $x^4 + 2\,mx^3 + m^2 x^2$, and $x^4 \begin{matrix} +a \\ +b \\ +2c \end{matrix}\ x^3 \begin{matrix} +ab \\ +2ac \\ +2bc \\ +c^2 \end{matrix}\ x^2 \begin{matrix} +2abc \\ +ac^2 \\ +bc^2 \end{matrix}\ x + abc^2$. That these products may differ only in the lowest term these three limitations are necessary. We must make first, $a + b + 2c = 2m$; secondly, $ab + 2ac + 2bc + c^2 = m^2$; and thirdly, $2abc + ac^2 + bc^2 = o$. From the two first equations we have, $a^2 + 2ab + b^2 + 4ac + 4bc + 4c^2$ $(= 4m^2) = 4ab + 8ac + 8bc + 4c^2 \because a - 2ab + b^2 = \overline{4a + 4b} \times c$. From the third equation, we have $2ab + ac + bc = o$, $\because c = - \dfrac{2ab}{a+b}$. This value of c being substituted in the equation $a^2 - 2ab + b^2 = \overline{4a + 4b} \times c$, there arises $a^2 - 2ab + b^2 - =$

$-8ab$, $\because a^2 = -6ba - b^2$ and $a = -3b \pm 2b\sqrt{2}$; ſo that either b, or a, will be embarraſſed with a ſurd.

Let the prime number whoſe logarithm is ſought by the application of the above products of four dimenſions, be denoted by $x+2$, if the number next greater is meaſured by 3; by $x+3$, if the number next leſs be meaſured by 3—and then all the other factors will be compoſite. For in the firſt caſe, $x-1$ will be meaſured by 2; x, by 3; $x+3$, by both 2 and 3; $x+5$, by 2; and $x+6$, by 3: In the other caſe, $x-1$, will be meaſured by 3; x, by 2; $x+2$, by both 2 and 3; $x+5$, by 3; and $x+6$, by 2.

If the prime number, whoſe logarithm is ſought, were denoted by any other factor than $x+2$, or $x+3$, other diviſors, beſides 2 and 3, ſhould come into conſideration; for in any caſe there would be at leaſt another factor, which, neither 2, nor 3, would meaſure.

Using then the notation directed above, the ratio of $x^2 \times \overline{x+5}|^2$ to, $\overline{x-1} \times \overline{x+2} \times \overline{x+3} \times \overline{x+6}$, of $\dfrac{x^2.\ \overline{x+5}^2}{\overline{x-1}.\ \overline{x+3}.\ \overline{x+6}}$

$: x+2$,

$: x + 2$, or of $\dfrac{x^2. \ \overline{x+5}|^2}{\overline{x-1}. \ \overline{x+2}. \ \overline{x+6}} : x + 3$, will always be a ratio of greater inequality.

In the application of thefe factors to the conftruction of logarithms $\dfrac{d}{s} = \dfrac{36}{2. \ x^2. \ \overline{x+5}|^2 - 36}$, which fraction (the above notation being ufed) when reduced to its loweft terms will have unity for its numerator. For, of x and $x + 5$, 2 always meafures one, and 3 the other, as appears from what has been faid above; therefore 4 meafures the fquare of one; and 9, the fquare of the other (the quote of two fquare numbers being equal to the fquare of the quote of their roots; which laft is in this cafe an integer) confequently 36 meafures the product of their fquares (the continual product of any factors being the fame, in whatever order the factors be taken) and therefore it alfo meafures double that product diminifhed by 36; i. e. $2. \ x^2. \ \overline{x+5}|^2 - 36$. Put the quote $\dfrac{2. \ x^2. \ \overline{x+5}|^2}{36} - 1 = y$; $\dfrac{d}{s}$ will then be equal to $\dfrac{1}{y}$, by fubftitution of which the expreffion of the general feries will be rendered more fimple as before.

THE

The investigation of products of five dimensions may be at present omitted.—For the difference of the products of five dimensions, which seem most commodious for the construction of logarithms, is so great as to destroy a considerable part of the advantage arising from the greatness of their sum.——Besides the very invention of these products, especially when the factors are large, is troublesome.—Not to mention the number of additions and subtractions necessary to find the logarithm required, after the logarithm of the ratio, as directed by the series, has been found.

This last objection indeed, would be of much less weight were products found whose difference should bear so small a ratio to their sum, as to preclude the necessity of using a single term of the series.

If $x \pm a$, $x \pm b$, $x \begin{smallmatrix} \mp a \\ \mp b \end{smallmatrix}$, denote the respective factors of two products; these products will be $x^3 * \begin{matrix} -a^2 \\ -ab. \\ -b^2 \end{matrix} x \mp ab. \overline{a+b}.$

Their arithmetical mean $x^3 * \begin{matrix} -a^2 \\ -ab. \\ -b^2 \end{matrix} x$, will be resolvable into three simple and rational factors; if $a^2 + ab + b^2$, be a perfect square.

ſquare. Make $a^2 + ab + b^2 = a^2 - 4ab + 4b^2$. Then will $5ab = 3b^2$ and $5a = 3b$. Let $b = 5 \because a = 3$. The products then with their reſpective factors will be $x^3 * - 49x - 120 = \overline{x+3} \times \overline{x+5} \times \overline{x-8}$; and $x^3 * - 49x + 120 = \overline{x-3}. \times \overline{x-5} \times \overline{x+8}$. And the factors of the arithmetical mean $x^3 - 49x$, will be $x - 7$, x, $x + 7$.

The ſquare of the arithmetical mean, $x^6 - 98\,x^4 + 2401\,x^2$, differing from the product of the extremes, $x^6 - 98\,x^4 + 2401\,x^2 - 14400$ by the abſolute term 14400. Here then we have two products of ſix dimenſions, which though they have a difference conſiderably greater, than the products already given of lower dimenſions, may yet be of ſome uſe in the conſtruction of logarithms.

Having ſhewn different methods of diminiſhing the value of the fraction $\frac{d}{s}$, in the application of the above general ſeries to the conſtruction of logarithms, by which means the ſeries will be made to converge faſter; I proceed now to an abbreviation of the ſeries itſelf, which will ſerve to compute the logarithms even of ſmall numbers to a much greater degree of accuracy, with ſcarce any increaſe of trouble.

If

If we put $y = \frac{s}{d}$, the ſeries may be expreſſed thus; $2p \times \frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5y^5} + \frac{1}{7y^7} +$ &c. which may be changed into the following equivalent one, $2p \times \frac{1}{y} + \frac{10}{30y^3 - 18y} + \frac{36}{1575y^7 - 2695y^5}$ — &c. whoſe two firſt terms are even more accurate thân the three firſt of the preceding—and its three firſt terms more accurate than five of that ſeries.

The ſecond and third terms of the above ſeries being reduced to a common denominator, their value is $\frac{5y^2 + 3}{15y^5}$. Upon dividing the denominator of this fraction by its numerator and taking the two firſt terms only of the quote for a new denominator, and unity for a numerator, we get the fraction $\frac{1}{3y^3 - \frac{9}{5}y}$ or $\frac{10}{30y^3 - 18y}$; which, reduced to an equivalent one whoſe denominator is $15y^5$, becomes $\frac{5y^2 + 3 + \frac{9}{5y^2} + \frac{27}{25y^4} + \frac{81}{125y^6} + \frac{243}{625y^8} + \&c.}{15y^5}$ the terms of the numerator decreaſing in the ratio of $5y^2 : 3$, i. e. in the ratio of the terms of the numerator of the fraction, which ariſes from the union of the ſecond and third terms of the ſeries.

Then

Then fince $\frac{10}{30y^3 - 18y} = \frac{5y^2+3}{15y^5} + \frac{3}{25y^7} + \frac{9}{125y^9} + \frac{27}{625y^{11}} + \frac{81}{3125y^{13}} +$ &c. we have $\frac{5y^2+3}{15y^5} = \frac{10}{30y^3 - 18y} - \frac{3}{25y^7} - \frac{9}{125y^9} - \frac{27}{625y^{11}} - \frac{81}{3125y^{13}} -$ &c. which being fubftituted for the fecond and third terms of the feries, we get the feries in this form, $2p \times \frac{1}{y} + \frac{10}{30y^3 - 18y} + \frac{4}{175y^7} + \frac{44}{1125y^9} + \frac{328}{6875y^{11}} + \frac{2072}{40625y^{13}} +$ &c.——Again; the terms $\frac{4}{175y^7}$ and $\frac{44}{1125y^9}$, (which are of the fame dimenfions with the fourth and fifth terms of the firft feries) being reduced to a common denominator, their value is $\frac{180y^2 + 308}{7875y^9}$; which, by a procefs fimilar to that already ufed, may be fhewn to be equivalent to, $\frac{36}{1575y^7 - 2695y^5} - \frac{3388}{50625y^{11}} - \frac{260876}{2278125y^{13}} -$ &c. the terms after the firft, decreafing in the ratio of $180y^2 : 308$, or of $45y^2 : 77$.

Then, by fubftitution, the feries comes out in this form: $2p \times \frac{1}{y} + \frac{10}{30y^3 - 18y} + \frac{36}{1575y^7 - 2695y^5} - \frac{428}{2275y^{11}} - \frac{75236}{1184625y^{13}} -$ &c.

THE two firſt terms of this ſeries may be uſed to very conſiderable advantage; with much the ſame trouble as the two firſt terms of the original ſeries; and with rather greater accuracy than the three firſt terms of that ſeries.

IF a third diviſion be required, this may be ſomewhat altered for the convenience of continual diviſions. Thus: $\frac{2p}{y} + \frac{10}{30y^2 - 18} . A + \frac{108}{\overline{9y^2 - 1} . \overline{y^2 - 1} . 175} . B$ — &c. the letters A, B, &c. denoting the terms as they ariſe.

THE three firſt terms of this ſeries are more accurate than three of the preceding; which likewiſe are more accurate than the five firſt terms of the original ſeries.

A FEW examples may ſerve to illuſtrate what has been ſaid.

NOTE, that Briggs's logarithm of 10 (which is 1,) divided by Napeir's logarithm of 10, (or, 2, 30258. 50929. 94045. 68401. 79914. 54684. 36420. 76011. 01488 &c.) the quote is, 43429. 44819. 03251. 82765. 11289. 18916. 60508. 22943. 97005 &c. $= p$. Then, ,86858. 89638. 06503. 65530. 22578. 37833. 21016. 45887.

45887. 94001 &c. $= 2p$. And, ,18861. 16970. 11613. 92922. 42603. $= p^2$.

Let it be required to find Briggs's logarithm of 101.

First, by Dr. Halley's method, or by the application of products of two dimensions.

$x-1$	x	$x+1$	Log. ratio of $\overline{x-1}.\overline{x+1}^{\frac{1}{2}} : x = \frac{p}{y}$ + &c. which is to be added to the logarithm of the antecedent, to find that of the consequent the ratio of $\overline{x-1}.\overline{x+1} : x^2$, being one of lesser inequality.
100	101	102	

$\overline{x-1}.\overline{x+1} = 10200$

$2.\overline{x-1}.\overline{x+1}+1 = 20401 = y$

20401) ,43429. 44819. 03251. 82765. 11289 (,00002. 12879. 01666. (74436. 68276 &c. $= \frac{p}{y}$.

$x - 1 = 100$	Log. 2,	
$x + 1 = 102$	2, 00860 01717 61917 56104 89366 92	
$\overline{x-1}.\overline{x+1}$	4, 00860 01717 61917 56104 89366 92	
$\overline{x-1}.\overline{x+1}^{\frac{1}{2}}$	2, 00430 00858 80958 78052 44683 46	
Ratio of, $\overline{x-1}.\overline{x+1}^{\frac{1}{2}} : x$	2 12879 01666 74436 68276 62	true to the 14th decimal place.
$x = 101$ Log.	2, 00432 13737 82625 52489 12960 08	true to the 14th place.

 Secondly

SECONDLY, by the application of products consisting of three factors.

$x-2$	$x-1$		$x+1$	$x+2$
101	102	*	104	105

$$\begin{array}{rr}
 & 104 \\ \hline
 & 52 \\
 & 52 \\ \hline
 & 104 \\
 & 260 \\ \hline
\frac{\overline{x+1}|^2}{4} = & 2704 \\
 & 101 \\ \hline
 & 2704 \\
 & 2704 \\ \hline
\frac{\overline{x+1}|^2 \times \overline{x-2}}{4} = & 273104
\end{array}$$

$$\frac{\overline{x+1}|^2 . \overline{x-2}}{2} + 1 = 546209 = y$$

Log. ratio of $\overline{x+1}|^2 \times \overline{x-2} : \overline{x-1}|^2 \times \overline{x+2} = \frac{2p}{y} +$ &c. Therefore, log. ratio of $x-2 : \overline{\frac{x-1}{x+1}}|^2 \times x+2 = \frac{2p}{y} +$ &c.

Which, as the ratio is one of lesser inequality, being subtracted from the logarithm of the consequent, gives the logarithm of the antecedent.

546209) ,86858 89638 06503 65530 22578 (,00000 15902 13569

$$(90914\ 40369 \text{ \&c.} = \frac{2p}{y}$$

$x-1 = 102$	Log.	2, 00860 01717 61917 56104 89366 92	
$x+1 = 104$		2, 01703 33392 98780 35484 77218 42	
$\frac{x-1}{x+1}$		—, 00843 31675 36862 79379 87851 50	
$\overline{\frac{x-1}{x+1}}\Big\vert^2$		—, 01686 63350 73725 58759 75703 00	
$x+2 = 105$		2, 02118 92990 69938 07279 35052 67	
$\overline{\frac{x-1}{x+1}}\Big\vert^2 \times x+2$		2, 00432 29639 96212 48519 59349 67	
Subtract log. ratio		15902 13569 90914 40369 94	true to the 18th decimal place.
$x-2 = 101$	Log.	2, 00432 13737 82642 57605 18979 73	true also to the 18th place.

THIRDLY,

THIRDLY, by the application of products of four dimensions.

$x-1$	x	$x+2$	$x+3$	$x+5$	$x+6$
98	99 *	101	102 *	104	105

$$\frac{x}{3} = \begin{array}{r} 33 \\ 33 \\ \hline 99 \\ 99 \\ \hline \end{array}$$

$$\frac{x^2}{9} = 1089$$

$$\frac{x+5}{2} = \begin{array}{r} 52 \\ 52 \\ \hline 104 \\ 260 \\ \hline \end{array}$$

$$\frac{\overline{x+5}|^2}{4} = \begin{array}{r} 2704 \\ 1089 \\ \hline 24336 \\ 21632 \\ 2704 \\ \hline \end{array}$$

$$\frac{x^2 \times \overline{x+5}|^2}{36} = 2944656$$

$$\frac{2.\, x^2 \times \overline{x+5}|^2}{36} - 1 = 5889311 = y$$

Log. ratio of $x^2.\overline{x+5}|^2 : x-1.x+2\ x+3.x+6$, or of $\frac{x^2\ \overline{x+5}|^2}{x-1.x+3\ x+6} : x+2 = \frac{2p}{y} +$ &c. Which is to be subtracted from the logarithm of the antecedent to find the logarithm of the consequent, the ratio being (as has been said above) one of greater inequality.

5889311)

5889311) ,86858. 89638. 06503. 65530. 22578 (,00000. 01474. 85667. 47561. 87362 &c. $= \frac{2p}{y}$

$x = 99$	Log.	1, 99563 51945 97549 91534 02557 77
$x + 5 = 104$		2, 01703 33392 98780 35484 77218 42
$x . \overline{x+5}$		4, 01266 85338 96330 27018 79776 19
$x^2 . \overline{x+5}^2$		8, 02533 70677 92660 54037 59552 39
$x - 1 = 98$		1, 99122 60756 92494 85663 81714 12
$x + 3 = 102$		2, 00860 01717 61917 56104 89366 92
$x + 6 = 105$		2, 02118 92990 69938 07279 35052 67
Ratio of $\frac{x^2 . \overline{x+5}^2}{\overline{x-1} . \overline{x+3} . \overline{x+6}} : x + 2$		1474 85667 47561 87362 67
		6, 02101 56940 10017 96609 93496 38
$x + 2 = 101$	Log.	2, 00432 13737 82642 57427 66056 01 true to the 21ft place.

The firft term of the feries, exhibiting the logarithm of the ratio true to the 21ft place.

If the fecond term of the feries, as it exifts in its contracted form, (viz. $\frac{10}{30y^2 - 18} . A$, or, $2p . \frac{10}{30y^3 - 18y}$) were ufed, we fhould obtain the logarithm of the ratio true, by the firft method, to the 32d decimal place; by the fecond to the 42d, and by the third method, to the 49th.

THE

The principal uſe of the factors of the products of ſix dimenſions in the conſtruction of logarithms, will appear, when (by reaſon of the greatneſs of the products which would ariſe) it becomes unneceſſary to uſe a ſingle term of the ſeries.

Thus; if it be ſufficient to have the logarithms of numbers true to 7 places of decimals, the logarithms of numbers exceeding 100 may be found by this method. Let the logarithm ſought, be that of 101, as before.

$x-8=86$	Log.	1, 93449 84512
$x-5=89$		1, 94939 00066
$x-3=91$		1, 95904 13923
$x+3=97$		1, 98677 17343
$x+5=99$		1, 99563 51946
$x+8=102$		2, 00860 01718
		11, 83393 69508
		5, 91696 84754
$x-7=87$		1, 93951 92526
$x\ \ =94$		1, 97312 78536
		3, 91264 71062
$x+7=101$	Log.	2, 00432 13692 true to the 8th decimal place.

The logarithm of the ratio of, $\overline{x-8}.\overline{x-5}.\overline{x-3}.\overline{x+3}.\overline{x+5}.\overline{x+8}:\overline{x-7}|^2.x^2.\overline{x+7}|^2=\frac{2p}{y}$ &c. Therefore the logarithm of the ratio of,

$$\frac{\overline{\overline{x-8}.\overline{x-5}.\overline{x-3}.\overline{x+3}.\overline{x+5}.\overline{x+8}}|^{\frac{1}{2}}}{\overline{x-7}.x}:x+7=\frac{p}{y}$$

&c. y being taken equal to $\frac{2.\overline{x-7}|^2.x^2.\overline{x+7}|^2}{14400}-1$; i. e. in the preſent inſtance 94755 506, 845. So that the logarithm

logarithm found will differ from the truth by lefs than half an unit in the 8th place of decimals.

HERE the number whofe logarithm is fought, being denoted by the factor $x+7$, the logarithm comes out lefs than the truth; if the number were denoted by the factor $x+8$, we fhould get the logarithm greater than the truth. The reafon of this is manifeft, from the relation between the products.

THE logarithms of numbers exceeding 300 may be found by this method to 10 places of decimals. And the logarithms of numbers exceeding 1300, to 14 places.

A SIMILAR application may be made of the factors of the products of four dimenfions, which will ferve to conftruct the logarithms of numbers exceeding 20,000 to fourteen places of decimals; fo that, if neceffary, we may have an eafy way of compleating the chiliads omitted by Briggs. E. G. Let it be propofed to find the logarithm of 19997, being even fomewhat lefs than 20000.

$x = 19995$

$x = 19995$	Log. 4, 30092 14084 6954
$x + 5 = 20000$	4, 30102 99956 6398
$x \times \overline{x+5}$	8, 60195 14041 3352
$x^2 \times \overline{x+5}\vert^2$	17, 20390 28082 6704
$x - 1 = 19994$	4, 30089 96877 7225
$x + 3 = 19998$	4, 30098 65640 4417
$x + 6 = 20001$	4, 30105 17098 4522
	12, 90293 79616 6164
$x + 2 = 19997$	Log. 4, 30096 48466 0540

The logarithm of the ratio of $x^2 . \overline{x+5}|^2 : \overline{x-1} . \overline{x+2} . \overline{x+3} . \overline{x+6} = \frac{2p}{y} +$ &c. But $y = \frac{2 . x^2 . \overline{x+5}|^2}{36} - 1$, i. e. in this inſtance 88844 44499 99999 9.—So that the logarithm of the ratio will be leſs than unit of the ſixteenth place of decimals.

The computation of the logarithms of ſmall numbers may be rendered more expeditious if, inſtead of attempting immediately to find the logarithm required, by the application of any of the preceding methods as they are above exemplified, we ſeek the logarithm of ſuch a multiple of the given number, as ſhall not have any of its prime factors greater than that number—with the ſame limitation alſo, on the numbers which are concerned in computing the logarithm of this multiple.

E. G. If it were propofed to find the logarithm of 31, the multiple by means of which it would be moft conveniently found by the application of products of two dimenfions, is according to Briggs, 341. Then, $17 \times 20 = 340 = x - 1$, $11 \times 31 = 341 = x$ and $18 \times 19 = 342 = x + 1$. Whence the logarithms of 340 and 342 being known from the logarithms of their prime factors, that of 341 may be found by Dr. Halley's method—from which the logarithm of 11 being fubtracted we get the logarithm of 31. Here $y = 2 \times 340 \times 342 + 1 = 232561$. So that ufing only the firft term of the feries we get the logarithm true to the 17th place. If we ufe the products of three dimenfions, the advantage will be confiderably greater.

The logarithm of 31 may be found from that of 899, which laft may be found by this method without knowing the logarithm of any prime number greater than 31. For $\overline{2}|^7 \times 7 = 896 = x - 2$; $3 \times 13 \times 23 = 897 = x - 1$; $29 \times 31 = 899 = x + 1$ and $\overline{2 \times 3 \times 5}|^2 = 900 = x + 2$. Here we have, $y = \frac{\overline{899}|^2 \times 896}{2} + 1 = 362074049$. The firft term of the feries gives the logarithm of the ratio true to the 27th place—without ufing even a fingle term of the feries, we find the logarithm true to the ninth decimal place.

If

If the logarithm of the number 7 be required it may be found from that of 28. Here $\overline{2}|^3 \times 3 = 24 = x - 2$, $\overline{5}|^2 = 25 = x - 1$, $\overline{3}|^3 = 27 = x + 1$, and $\overline{2}|^2 \times 7 = 28 = x + 2$. And $y = \frac{27^2 \times 24}{2} + 1 = 8749$. If we uſe the two firſt terms of the ſeries as it was firſt laid down, we may find the logarithm to 19 places of decimals. But if we uſe the two firſt terms of the contracted ſeries, which may be done with only the additional trouble ariſing from diminiſhing the ſecond diviſion by 1, 8, we may get the logarithm true to 28 places.

It is manifeſt that the logarithm of ſome number muſt be found by the immediate application of the ſeries. If then Napier's logarithm of $1\frac{1}{4}$ be found by this method, we thence obtain the logarithm of $\frac{125}{64}$. The logarithm of $\frac{128}{125}$ being found in the ſame manner, and added to that of $\frac{125}{64}$, gives Napier's logarithm of 2. If the logarithms be required of Briggs's form, having found Napier's logarithm of 8, from his logarithm of 2, by adding the logarithm of $1\frac{1}{4}$ as already found, we get Napier's logarithm of 10; the reciprocal of which will be the modulus of

Briggs's ſyſtem, repreſented by p in the ſeries; by which if Napier's logarithms of $\frac{5}{4}$, and $\frac{128}{125}$ be multiplied we ſhall get Briggs's logarithms of the ſame numbers. If then we ſubtract Briggs's logarithm of $\frac{5}{4}$, from his logarithm of 10, and divide the remainder by 3; or if we add Briggs's logarithm of $\frac{128}{125}$ to his logarithm of 1000, and divide the ſum by 10; we ſhall have Briggs's logarithm of 2.—The logarithm of 2 being found, the logarithms of all other numbers may have their computation facilitated by ſome one or other of the preceding methods of increaſing the convergency of the ſeries.

As a due application of the binomial theorem furniſhes us with a ſeries for finding the logarithm of any natural number, (which is the ſame with the logarithm of the ratio of that number to unity) ſo likewiſe it may be ſhewn from the ſame principles that the ſum of the ſeries $1 + \frac{l}{1} + \frac{l^2}{1.2} + \frac{l^3}{1.2.3} + \frac{l^4}{1.2.3.4} +$ &c. will be the natural number correſponding to Napier's logarithm

l; and

l; and that the ſum of the ſeries $1 - \frac{l}{1} + \frac{l^2}{1.2} - \frac{l^3}{1.2.3} + \frac{l^4}{1.2.3.4} -$ &c. will be the natural number correſponding to Napier's logarithm $-l$.

If l denote Napier's logarithm of the improper fraction $\frac{b}{a}$, then will $-l$ be Napier's logarithm of its reciprocal $\frac{a}{b}$.—— Therefore,

$$\frac{b}{a} = 1 + l + \frac{l^2}{2} + \frac{l^3}{6} + \frac{l^4}{24} + \frac{l^5}{120} + \frac{l^6}{720} + \frac{l^7}{5040} + \frac{l^8}{40320} + \&c.$$

$$\frac{a}{b} = 1 - l + \frac{l^2}{2} - \frac{l^3}{6} + \frac{l^4}{24} - \frac{l^5}{120} + \frac{l^6}{720} - \frac{l^7}{5040} + \frac{l^8}{40320} - \&c.$$

$$\frac{b^2 - a^2}{ab} = 2l + \frac{l^3}{3} + \frac{l^5}{60} + \frac{l^7}{2520} + \&c.$$

$$\frac{b^2 + a^2}{ab} = 2 + l^2 + \frac{l^4}{12} + \frac{l^6}{360} + \frac{l^8}{20160} + \&c.$$

$$\frac{b^2 + a^2}{ab} - 2, \left(= \frac{b^2 - 2ba + a^2}{ab}\right) = l^2 + \frac{l^4}{12} + \frac{l^6}{360} + \frac{l^8}{20160} + \&c.$$

This ſeries being divided by the ſeries $2l + \frac{l^3}{3} + \frac{l^5}{60} + \frac{l^7}{2520} +$ &c. the quote will be the ſeries, $\frac{l}{2} - \frac{l^3}{24} + \frac{l^5}{240} - \frac{17 l^7}{40320} +$ &c.

Call

Call the ſum of this ſeries, s. Then, $\frac{b-a}{b+a}$ $(=\frac{b^2-2ba+a^2}{ab} \div \frac{b^2-a^2}{ab}) = s \because b-a = sb+sa,\ b-sb = a+sa,\ b = \frac{1+s}{1-s}.\ a,$ and $a = \frac{1-s}{1+s}.\ b.$

Here then we have a ſeries conſiſting of but half the number of terms of either of the ſeries firſt propoſed; but a diviſion is afterwards requiſite to find the natural number $\frac{b}{a}$, or $\frac{a}{b}$.

Let us endeavour to render this ſeries more convenient for practice and apply it to find the natural numbers of logarithms of any ſyſtem whatever, without the trouble of reducing the given logarithm to Napier's ſyſtem.——If we aſſume the firſt term for the whole ſeries, i. e. $\frac{l}{2} = s$; then will $\frac{b}{a} = \frac{1+\frac{l}{2}}{1-\frac{l}{2}}$ and $\frac{a}{b} = \frac{1-\frac{l}{2}}{1+\frac{l}{2}}.$ making L to denote the logarithm of $\frac{b}{a}$ in any ſyſtem

whatſoever,

whatſoever, ſince $\frac{L}{l}=p$, $\frac{L}{p}=l$; then by ſubſtitution, $\frac{b}{a}=\frac{2p+L}{2p-L}$ and $\frac{a}{b}=\frac{2p-L}{2p+L}$.

This abbreviation Dr. Halley has deduced from contracting into one two terms of the firſt ſeries.

If we unite the two firſt terms of the ſeries $\frac{l}{2}-\frac{l^3}{24}+\frac{l^5}{240}-\frac{17\,l^7}{40320}+$ &c. we may obtain a method ſtill more accurate, which however requires a ſecond operation; but yet perhaps may be preferable to the application of the ſeries in its original form.— The terms $\frac{l}{2}$ and $-\frac{l^3}{24}$ being reduced to a common denominator, their value is $\frac{12\,l-l^3}{24}$; which, by a proceſs ſimilar to that which has been uſed above, will appear to be equal to $\frac{6\,l}{12+l^2}-\frac{l^5}{288}+\frac{l^7}{3456}-$ &c. Then by ſubſtitution the ſeries becomes $\frac{6\,l}{12+l^2}+\frac{l^5}{1440}-\frac{l^7}{7560}+$ &c. It appears then, that the term $\frac{6\,l}{12+l^2}$ will more accurately repreſent the entire ſeries than the two terms $\frac{l}{2}-\frac{l^3}{24}$.

Assume

Aſsume $\frac{6l}{12+l^2} = s$. Then $\frac{b}{a}$ $\left(= \overline{1 + \frac{6l}{12+l^2}} \div \overline{1 - \frac{6l}{12+l^2}}\right)$ $= \frac{12+6l+l^2}{12-6l+l^2}$; and $\frac{a}{b} = \frac{12-6l+l^2}{12+6l+l^2}$. In any ſyſtem whatſoever $\frac{b}{a} = \frac{12p^2+6pL+L^2}{12p^2-6pL+L^2}$ and $\frac{a}{b} = \frac{12p^2-6pL+L^2}{12p^2+6pL+L^2}$. Make $\frac{12p^2}{L} = q$; then $\frac{b}{a} = \frac{q+6p+L}{q-6p+L}$ and $\frac{a}{b} = \frac{q-6p+L}{q+6p+L}$.

The fraction $\frac{12+6l+l^2}{12-6l+l^2}$ thrown into a ſeries will be $1 + l + \frac{l^2}{2} + \frac{l^3}{6} + \frac{l^4}{24} + \frac{l^5}{144} +$ &c. ſo that the natural number correſponding to Napier's logarithm l will be accurately expreſſed thus $\frac{12+6l+l^2}{12-6l+l^2} + \frac{l^5}{720} +$ &c. Whence it appears that the fraction $\frac{12+6l+l^2}{12-6l+l^2}$ will more accurately expreſs the natural number of Napier's logarithm l, than the five firſt terms $\left(1 + l + \frac{l^2}{2} + \frac{l^3}{6} + \frac{l^4}{24}\right)$ of the original ſeries. Now in the application of theſe terms to find the natural number of a logarithm, beſides reducing the given logarithm to Napier's ſyſtem if it be of any other, three multiplications are neceſſary, not to mention the

diviſions

divifions performed by fmall divifors. Whereas, to find the natural number $\frac{b}{a}$, by the application of the fraction $\frac{q+6p+L}{q-6p+L}$, or the natural number $\frac{a}{b}$, by the application of the fraction $\frac{q-6p+L}{q+6p+L}$, two divifions are fufficient, of whatever fyftem the logarithm may be. For $12p^2$ being given (fince p is the modulus of the fyftem) q is found by one divifion. Another divifion will give the value of the fraction $\frac{q\pm 6p+L}{q\mp 6p+L}$, all its terms being known and of one dimenfion.

SINCE $\frac{b}{a}=\frac{q+6p+L}{q-6p+L}$ and $\frac{a}{b}=\frac{q-6p+L}{q+6p+L}$, we have $b=a+\frac{12pa}{q-6p+L}$ and $a=b-\frac{12pb}{q+6p+L}$.

WHENCE if a logarithm of any fyftem be given, to find its correfponding natural number; find the difference of the given logarithm and the neareft logarithm to it (in the fame fyftem) whofe natural number is known. Call this difference L; the modulus of the fyftem p; and $\frac{12p}{L}=q$. Then, if the given

logarithm be the greater, its natural number will be the natural number of the aſſumed logarithm increaſed by the product of the ſame number into the fraction $\frac{12p}{q-6p+L}$; if leſs, its natural number will be the natural number of the aſſumed logarithm, diminiſhed by the product of the ſame number into the fraction $\frac{12p}{q+6p+L}$. This is not perfectly accurate; but will be a nearer approximation to the truth, the leſs the difference is of the given and the aſſumed logarithm.